Generation of 7x7 Calcrostic Puzzles and Rational Solution of Underdetermined Polynomial Systems

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Abstract

This poster reports on the generation of new types of mathematical 'calcrostic' puzzles. In these puzzles the task is to determine for each letter the corresponding digit that the letter represents such that a number of conditions arranged in the form of a crossword problem are satisfied. Earlier versions of these puzzles were used in Caribou Mathematics Contests with large Canadian and international participation.

Through our work we are able to generate larger calcrostics, now of size 5x5 and 7x7 instead of 3x3; more complicated ones, now involving rational numbers rather than integers, and puzzles that require many more conditions to be satisfied through extra diagonal relations.

Apart from many new procedures that had to be written for handling these new types of puzzles, the key problem was that much larger systems of under-determined polynomial equations needed to be solved with computer algebra. We developed a method of computing special solutions of highly under-determined and highly non-linear, polynomial equations. The new technique was implemented and merged into the computer algebra package CRACK.

Introduction

The word *Calcrostic* is coined from showing *CALculations* in form of *aCROSTIC* word puzzles. An example of such a problem with the corresponding solution is shown below:

EDKH	÷	KF	=	AA	1320	÷	24	=	55
$-$	+	+		+	$-$	+	+		+
EDB	×	J	=	EHCG	137	×	8	=	1096
=	=	=		=	=	=	=	=	=
EEJD	$-$	DK	=	EEAE	1183	$-$	32	=	1151

Each letter represents a digit (one of 0,1,...,9). Equal letters represent equal digits, different letters represent different digits. For example, EDKH represents a 4-digit integer with four different digits. The challenge is to find the value of each letter such that all the 3 horizontal and 3 vertical equations are satisfied.

In the above example, EDKH and KF represent the numbers 1320 and 24 respectively. We can see that E stands for 1, D for 3, K for 2, H for 0, and so on. In our recent work we extended earlier versions of the program generating integer 3×3 puzzles like the one above to now generate also 5×5 and 7×7 with integer *and* rational numbers, see the 7×7 on the right.

Although 7×7 puzzles are harder to solve (for a solution, see Interactive Caribou Puzzle Solving by Hill, M., on *https://test.cariboutests.com/test game/calcrostic.php)*, the reward when finding the value of the last letter is higher (36 conditions become satisfied: 7 horizontal, 7 vertical, all 11 diagonals running from the top left to the bottom right and all 11 diagonals running from the top right to the bottom left evaluate to zero).

The computational challenge of this problem is to find rational solutions of the underdetermined system of 36 equations for 49 unknowns, u_i as in the algorithmic work section.

Example

An example of a generated problem:

$\frac{cgj}{af}$	$+$	$\frac{-ii}{j}$	$+$	$\frac{-edhd}{ahf}$	$-$	i	$-$	f	$\times$	e	$-$	$\frac{-fca}{eb}$
$-$	$-$	$\div$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
$\frac{-ii}{j}$	$-$	h	$+$	i	$+$	aa	$-$	c	$-$	$\frac{hi}{j}$	$-$	$-e$
$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
$\frac{hgca}{ahf}$	$+$	i	$+$	f	$-$	$\frac{a}{f}$	$-$	$\frac{cbjj}{ahf}$	$-$	j	$+$	$\frac{ih}{j}$
$+$	$+$	$-$	$-$	$-$	$+$	$\times$	$+$	$+$	$-$	$+$	$-$	$+$
i	$+$	$\frac{cfih}{ahf}$	$-$	$\frac{eij}{ahf}$	$+$	$\frac{-hhed}{eb}$	$+$	$\frac{ihab}{ebg}$	$-$	$\frac{-hfca}{fc}$	$-$	ah
$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$
$\frac{-cbai}{ahf}$	$-$	$\frac{dfeb}{ahf}$	$-$	$\frac{bjei}{ahf}$	$+$	$\frac{caeh}{fc}$	$+$	$\frac{eadc}{cf}$	$-$	$\frac{-cbcb}{ahf}$	$\times$	$\frac{a}{e}$
$+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$\div$	$-$	$+$	$+$	$-$
$\frac{agja}{eb}$	$+$	$\frac{achf}{eb}$	$-$	$\frac{bja}{ahf}$	$-$	$\frac{eejb}{ebg}$	$-$	$\frac{agie}{cf}$	$+$	$\frac{-hgc}{cf}$	$+$	$\frac{-iba}{ai}$
$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
$\frac{-aheah}{ahf}$	$+$	$\frac{agja}{eb}$	$+$	$\frac{ja}{f}$	$-$	$\frac{-dehi}{ahf}$	$-$	$\frac{fdh}{jg}$	$-$	$\frac{-iba}{ai}$	$+$	$\frac{-ahef}{eb}$

The corresponding solution:

$\frac{409}{15}$	$+$	$\frac{-88}{9}$	$+$	$\frac{-2636}{135}$	$-$	8	$-$	5	$\times$	2	$-$	$\frac{-541}{27}$
$-$	$-$	$\div$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
$\frac{-88}{9}$	$-$	3	$+$	8	$+$	11	$-$	4	$-$	$\frac{38}{9}$	$-$	-2
$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
$\frac{3041}{135}$	$+$	8	$+$	5	$-$	$\frac{1}{5}$	$-$	$\frac{4799}{135}$	$-$	9	$+$	$\frac{83}{9}$
$+$	$+$	$-$	$-$	$-$	$+$	$\times$	$+$	$+$	$-$	$+$	$-$	$+$
8	$+$	$\frac{4583}{135}$	$-$	$\frac{289}{135}$	$+$	$\frac{-3326}{27}$	$+$	$\frac{8317}{270}$	$-$	$\frac{-3541}{54}$	$-$	13
$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$
$\frac{-4718}{135}$	$-$	$\frac{6527}{135}$	$-$	$\frac{7928}{135}$	$+$	$\frac{4123}{54}$	$+$	$\frac{2164}{45}$	$-$	$\frac{-4747}{135}$	$\times$	$\frac{1}{2}$
$+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$\div$	$-$	$+$	$+$	$-$
$\frac{1091}{27}$	$+$	$\frac{1435}{27}$	$-$	$\frac{791}{135}$	$-$	$\frac{2297}{270}$	$-$	$\frac{1082}{45}$	$+$	$\frac{-304}{45}$	$+$	$\frac{-871}{18}$
$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$
$\frac{-13213}{135}$	$+$	$\frac{1091}{27}$	$+$	$\frac{91}{5}$	$-$	$\frac{-6238}{135}$	$-$	$\frac{563}{90}$	$-$	$\frac{-871}{18}$	$+$	$\frac{-1325}{27}$

Generation of Puzzles

To create, for example, such a 7×7 puzzle, one

1. fills the $(7+6)\times(7+6)$ matrix with randomly selected operators $+,-,\times,\div$, and with 7×7 variables $u_i,\ \ i=1...49$ (not sequences of letters representing digits like in the introduction above),
2. formulates the under-determined system of $7+7+11+11=36$ polynomial equations for the 49 variables u_i,
3. computes rational solutions for this system with possibly several free parameters,
4. repeats the following steps:
   - replaces free parameters by random integer numbers, so that divisors and denominators do not become zero,
   - encodes digits by letters,
   - determines all solutions of the resulting puzzle

   until the created puzzle has only one solution.

Algorithmic Work

- Developed three partial splitting methods, one of which is to split an unsolved equation
$$0=P(u_j)=\sum_{n=0}^{d_i}A_{in}(u_k)u_i^n \quad \text{into}$$
$0=A_{in}$ for n = 2,..., d_i and
$0=A_{i0}+A_{i1}u_i$
- found heuristics to pick an optimal equation $P(u_j)$ and variable u_i for the splitting, and
- incorporated a corresponding module into CRACK to create a solver for underdetermined highly non-linear polynomial equations and systems.

References

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