Colouring of the First 313230 Knots

Kindeep Singh Kargil‡ | Lechuan (Sibelius) Peng† | Saad Ali Ghaznavi† | Thomas Wolf‡

‡Brock University, †University of Waterloo

Introduction

A knot diagram is a 3-dimensional closed knot-line projected on 2 dimensions. If two diagrams can be deformed into each other by stretching, shortening or shifting the line without cutting it, then both diagrams represent the same mathematical knot. In order to determine whether this is the case, knot invariants can be computed for each diagram. As the name indicates, the value of each invariant is unaffected by deformations. This means that even if only one invariant has different values for two diagrams, then both diagrams can never be deformed into each other, so they must represent different mathematical knots.

The easiest to determine invariant is the Yes/No property which states whether the diagram (and thus the knot) is 3-colourable or not.

A knot diagram is called 3-colourable, if

1. each strand (part of the knot-line that can be drawn without lifting the pen) is coloured with one colour,
2. at least 2 colours are used,
3. on each crossing all three meeting strands have either the same colour or all three have different colours.

Examples of 3-colourable knots are $3_1, 6_1$ and 7_4.

Knots that are not 3-colourable may be 5-, 7-, ..., n-colourable (n prime). A knot is called 'Fox n-colourable' ([1],[2]) if the first two conditions of 3-colourability and a modified third condition hold.

$3'$. Each colour is represented by a number in $0, 1, ..(n-1)$ and at each crossing, the number assigned to the "above-strand" is the average of the numbers assigned to the colours of the other two "below-strands" modulo n.

In addition to the Yes/No property of n-colourability for each prime number n, the total number of n-colourings is also an invariant.

3-colourability is a very weak invariant because it partitions all knots into just 2 groups according to whether they are 3-colourable or not. The invariant becomes stronger when adding all, n-colourability and number of n-colourings for all prime integer n, into one combined colouring invariant. For example, knot 9_{37} has the combined colour invariant $3^3, 5^2$ whereas knot 9_{40} has the combined colour invariant $3^2, 5^3$ showing that both diagrams represent different mathematical knots.

Display

The poster shows diagrams of all 250 knots with maximally 10 crossings.

If a knot has at least one n-colouring for prime n then one colouring with minimal n is shown. Otherwise, the diagram has only one colour (Blue).

A caption $A_B : c^d, e^f, ...$ indicates that the B^{th} knot with A crossings in the Rolfson table of knots [3] has c^d many c-colourings and e^f many e-colourings, ... In other words, d is a measure of the degeneracy of the coefficient matrix of the linear algebraic system mod c which results from colouring condition $3'$.

Results

Although the invariance of knot colouring is known for many years, a colouring classification of knots is so far not available in databases about knots, like the KnotInfo database [4].

By combining n-colourability and the number of n-colourings for all n as a combined invariant, we obtain the following table with the columns holding: the crossing number c, number k_c of knots with crossing number c, number I_c of different combined colour invariants, and the average number of knots sharing the same combined colour invariants.

Table 1: Table of number of combined colour invariants

c	k_c	I_c	k_c/I_c	$I_c^{1/(c-3)}$
3	1	1	1	
4	1	1	1	1.000
5	2	2	1	1.414
6	3	3	1	1.442
7	7	7	1	1.627
8	21	14	1.5	1.695
9	49	29	1.69	1.753
10	165	53	3.11	1.763
11	552	93	5.94	1.762
12	2176	162	13.43	1.760
13	9988	271	36.86	1.751
14	46972	488	96.25	1.756
15	253293	855	296.25	1.755

Knots may allow very different numbers of n-colourings. For example, every knot allows 3^1 (trivial) 3-colourings where each strand has the same colour. Knot 8_{18} allows 3^2 3-colourings, knot 9_{35} has 3^3 3-colourings and knot 12_{n553} does even allow 3^4 3-colourings. Knots with $\geq n^3$ n-colourings often show a high degree of symmetry. For example, in the case of 8_{18}, the differences of successive numbers in the Dowker encoding are 5,5,5,5,5,5,5,5.

Knots 10_{120} and 10_{122} have colourings for 3 different n. The survey identifies knots $10_{124}, 10_{153}$ not to be n-colourable for prime $n < 1000$ which then turn out also not to be colourable for $n < 100,000$ and by a theoretical argument they are not colourable for larger n either. The growth of the number of colour groups by a factor of about 1.75 from one crossing number to the next is remarkably constant.

Table 2: Table of $B(c) = n_{\max}^{1/(c-1)}$

c	$n_{\max}$	knot	$B(c)$
3	3	3_1	1.732
4	5	4_1	1.710
5	7	5_2	1.627
6	13	6_3	1.670
7	19	7_6	1.634
8	37	8_{17}	1.675
9	61	9_{33}	1.672
10	109	10_{115}	1.684
11	199	11_{a301}	1.698
12	353	12_{a1188}	1.705
13	593	13_{a4620}	1.703
14	1103	14_{a16476}	1.714
15	1823	15_{a65606}	1.710

An unexpected and very useful result is a simple approximate formula for the highest n value $n_{\max}$ that allows n-colourability in dependence of the crossing number c. It is $n_{\max}(c) \approx 1.7^{c-1}$. The benefit of such a formula is to check n-colourability only for n up to $n_{\max}$ when $n_{\max}$ is exactly known and to check n up to $n = 1.75^{c-1}$ for $c > 15$ when we do not have the exact value of $n_{\max}$. Here 1.75 instead of 1.7 adds safety.

In general the computer program is very fast. It determines all n-colourings for prime $n < 1000$ for the first 250 knots on a single 3.7 GHz CPU computer in less than 3 sec and using $n_{\max}$ in < 1 sec. For the first 313230 knots with up to 15 crossings the computation would take over a month on a single CPU. We parallelized the computation.

For single knots of same size computation times may vary widely. For example, for 15_{n76000} it takes 0.39 sec but for 15_{a81645} it takes 2:19 min so over 350 times longer. The speedups bring more when the maximal number of consecutive over- and underpasses in a knot is larger.

The complete colourability classification for knots with up to 15 crossings is available at [5]. The module used to perform the computations is part of a freely available interactive workbench for knots running under linux [6].

References

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